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WAVE PROPAGATION IN THE CROSS-SHAPED CONNECTION OF INFINITE ELASTIC STRIPS*

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A method of solving dynamic contact problems for rigidly coupled elastic bodies is proposed. The problem of the travelling of harmonic Rayleigh-Lamb waves arriving from infinity at a cross-shaped connection between elastic strips is considered. The contact dynamic interaction between two elastic bodies when there is no friction was previously investigated in /1-3/.

Suppose a horizontal strip with constants ρ_1, λ_1, μ_1 occupies a region $|x| < \infty$, $-2H \leqslant y \leqslant 0$ in a Cartesian system of coordinates. The two vertical half-strips with constants ρ_2, λ_2, μ_2, occuping the regions $|x| \leqslant 1$, $0 \leqslant y < \infty$ and $|x| \leqslant 1$, $-\infty < y \leqslant -2H$ are rigidly coupled with a horizontal strip in the contact region. All the geometrical parameters are dimensionless. A symmetrical Rayleigh-Lamb wave propagates in the horizontal strip in the positive direction of the x axis.

In view of the symmetry of the problem about the mean line of the horizontal strip it is sufficient to consider the upper half of the mechanical system. The problem can also be split into symmetrical and antisymmetrical parts relative to the y axis. The symmetrical part will be considered below; the antisymmetrical part can be considered in a similar manner.

The boundary conditions for the horizontal strip have the form

$$\begin{gathered} v_1(x, -H) = \tau_1(x, -H) = 0, \ |x| < \infty \\ \sigma_{y1}(x, 0) = \tau_1(x, 0) = 0, \ 1 < |x| < \infty \\ \sigma_{y1}(x, 0) = p(x), \ \tau_1(x, 0) = g(x), \ |x| \leqslant 1 \end{gathered} \tag{1}$$

where $v_1(x, y)$ is the displacement along the y axis, $\sigma_{y1}(x, y)$ and $\tau_1(x, y)$ are the normal and tangential stresses, and $p(x)$ and $g(x)$ are the components of the unknown contact stresses; the harmonic factor $e^{-i\omega t}$ is omitted here and henceforth.

The travelling wave is specified as a propagating Rayleigh-Lamb wave mode, and corresponds to the uniform solution of the Lame equations for a horizontal strip without a load on the side faces. The displacements in the specified wave have the form

$$u_0(x, y) = -A\alpha_0 (k_1^\circ)^{-1} \Phi_1(\alpha_0, y) \sin \alpha_0 x, \quad v_0(x, y) = A\Phi_2(\alpha_0, y) \cos \alpha_0 x$$

$$\Phi_1(\alpha_0, y) = (2\alpha_0^2 - \varkappa_2^2) \operatorname{sh} k_2^\circ H \operatorname{ch} k_1^\circ (y + H) - 2k_1^\circ k_2^\circ \operatorname{sh} k_1^\circ H \operatorname{ch} k_2^\circ (y + H)$$

$$\Phi_2(\alpha_0, y) = (2\alpha_0^2 - \varkappa_2^2) \operatorname{sh} k_2^\circ H \operatorname{sh} k_1^\circ (y + H) - 2\alpha_0^2 \operatorname{sh} k_1^\circ H \operatorname{sh} k_2^\circ (y + H)$$

$$k_1^\circ = (\alpha_0^2 - \varkappa_1^2)^{1/2}, \quad k_2^\circ = (\alpha_0^2 - \varkappa_2^2)^{1/2}, \quad \varkappa_1^2 = \frac{\rho_1 \omega^2}{\lambda_1 + 2\mu_1},$$

$$\varkappa_2^2 = \frac{\rho_1 \omega^2}{\mu_1}$$

Here A is the amplitude and α_0 is the real root of the Rayleigh-Lamb equation

$$\begin{gathered} \Delta(\alpha) \equiv (2\alpha^2 - \varkappa_2^2)^2 \operatorname{sh} k_2 H \operatorname{ch} k_1 H - 4\alpha^2 k_1 k_2 \operatorname{sh} k_1 H \operatorname{ch} k_2 H = 0 \\ k_1^2 = \alpha^2 - \varkappa_1^2, \ k_2^2 = \alpha^2 - \varkappa_2^2 \end{gathered} \tag{2}$$

For the vertical half-strip the side faces are free from stresses. In the contact region the conditions of rigid coupling are satisfied. The solution of boundary value problem (1) is found by an integral Fourier transformation with respect to the x coordinate (α is the conversion parameter). We obtain for the displacements of the horizontal strip

$$u_1(x, y) = \frac{1}{\mu_1 \sqrt{2\pi}} \int_{\delta} [-p(\alpha) \alpha \Phi_1(\alpha, y) + ik_2 g(\alpha) \Phi_3(\alpha, y)] \frac{\sin \alpha x}{\Delta(\alpha)} d\alpha + u_0(x, y), \tag{3}$$

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$$v_1(x, y) = \frac{1}{\mu_1\sqrt{2\pi}} \int_\Omega [k_1 p(\alpha)\Phi_3(\alpha, y) +$$
$$i\alpha g(\alpha)\Phi_4(\alpha, y)] \frac{\cos\alpha x}{\Delta(\alpha)} d\alpha + v_0(x, y)$$
$$\Phi_3(\alpha, y) = (2\alpha^2 - \varkappa_2^2)\operatorname{ch} k_1 H \operatorname{ch} k_2(y + H) -$$
$$2\alpha^2 \operatorname{ch} k_2 H \operatorname{ch} k_1(y + H)$$
$$\Phi_4(\alpha, y) = 2k_1k_2 \operatorname{ch} k_2 H \operatorname{sh} k_1(y + H) - (2\alpha^2 -$$
$$\varkappa_2^2)\operatorname{ch} k_1 H \operatorname{sh} k_2(y + H)$$
$$p(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} p(x)\cos\alpha x\, dx, \quad g(\alpha) = -\frac{i}{\sqrt{2\pi}} \int_{-1}^{1} g(x)\sin\alpha x\, dx$$

The contour of integration in (3) coincides with the real axis, circumventing the singularities of the integrands in accordance with the limit-absorption principle /4/.

For the vertical semistrip the solution is constructed in the form of series in the uniform solutions for an infinite strip with boundaries free from stresses

$$u_2(x, y) = \sum_{n=1}^{\infty} C_n u_n(x)\exp(i\beta_n y) \qquad (4)$$
$$v_2(x, y) = \sum_{n=1}^{\infty} C_n v_n(x)\exp(i\beta_n y)$$
$$v_n(x) = i\,\frac{2\beta_n \operatorname{ch} m_{1n}x \operatorname{ch} m_{2n} - (\beta_n^2 + m_{2n}^2)\operatorname{ch} m_{2n}x \operatorname{ch} m_{1n}}{m_{1n}\beta_n \operatorname{ch} m_{2n}}$$
$$u_n(x) = 2\,\frac{(\beta_n^2 + m_{2n}^2)\operatorname{sh} m_{1n}x \operatorname{sh} m_{2n} - 2\beta_n^2 \operatorname{sh} m_{2n}x \operatorname{sh} m_{1n}}{(\beta_n^2 + m_{2n}^2)\operatorname{sh} m_{2n}}$$
$$m_{1n}^2 = \beta_n^2 - \delta_1^2, \quad m_{2n}^2 = \beta_n^2 - \delta_2^2, \quad \delta_1 = \frac{\rho_2\omega^2}{\lambda_2 + 2\mu_2}, \quad \delta_2 = \frac{\rho_2\omega^2}{\mu_2}$$

Here C_n are unknown complex constants, and β_n are the roots of the Rayleigh-Lamb dispersion equation for an infinite strip with constants ρ_2, μ_2, λ_2

$$(2\beta^2 - \delta_2^2)^2 \operatorname{ch} m_1 \operatorname{sh} m_2 - 4\beta^2 m_1 m_2 \operatorname{ch} m_2 \operatorname{sh} m_1 = 0 \qquad (5)$$

Eq.(5) has a finite number of real and imaginary roots and a denumerable set of complex roots for each value of the frequency. The real and imaginary roots lie symmetrically about the origin of coordinates, while the complex roots are distributed symmetrically in all four quadrants of the complex plane. The summation in (4) is carried out over the roots lying in the upper half-plane, taking into account the requirements of the principle of energy radiation /4/.

From the condition for the stresses in the contact region to be equal we can obtain

$$p(x) = \sum_{n=1}^{\infty} C_n\sigma_{yn}(x), \quad g(x) = \sum_{n=1}^{\infty} C_n\tau_n(x) \qquad (6)$$

Here $\sigma_{yn}(x)$ and $\tau_n(x)$ are the stresses corresponding to the uniform solutions

$$\sigma_{yn}(x) = 2\mu_2(m_{1n}\operatorname{ch} m_{2n})^{-1}[(2\beta_n^2 - \delta_2^2)\operatorname{ch} m_{1n}\operatorname{ch} m_{2n}x -$$
$$(2m_{1n}^2 + \delta_2^2)\operatorname{ch} m_{2n}\operatorname{ch} m_{1n}x]$$
$$\tau_n(x) = 4\mu_2 i\beta_n(\operatorname{sh} m_{2n})^{-1}(\operatorname{sh} m_{2n}\operatorname{sh} m_{1n}x - \operatorname{sh} m_{1n}\operatorname{sh} m_{2n}x)$$

Substituting relation (6) into solution (3) we can represent the displacement of the horizontal strip $u_1(x, y)$ and $v_1(x, y)$ in terms of the uniform solutions of the vertical strip. To obtain an algebraic set of equations in C_n we will use the Reissner variational principle, which in the case considered, taking (6) into account, has the form

$$\sum_{n=1}^{\infty} C_n \int_{-1}^{1} [\sigma_{yj}(x)v_n(x) + \tau_j(x)u_n(x)]\,dx = \qquad (7)$$
$$\int_{-1}^{1} [\sigma_{yj}(x)v_1(x, 0) + \tau_j(x)u_1(x, 0)]\,dx, \quad j = 1, 2, \ldots$$

Here $v_1(x, 0)$ and $u_1(x, 0)$ are the displacements in the coupling region, obtained from (3). The uniform solutions occurring in (7) satisfy the conditions of generalized orthogonality /5/

$$W_{nj} = \int_{-1}^{1} [v_n(x)\sigma_{yj}(x) - u_j(x)\tau_n(x)]\,dx = 0, \quad \beta_n^2 \neq \beta_j^2 \tag{8}$$
$$W_{nn} \neq 0, \; \beta_n^2 = \beta_j^2; \; j, n = 1, 2, \ldots$$

Using (8), we obtain from (7) an infinite algebraic set of equations in C_n of the following form:

$$C_j S_j + \sum_{n=1}^{\infty} C_n a_{nj} = b_j, \quad j = 1, 2, \ldots; \quad S_j = \frac{W_{jj}}{\mu_2} \tag{9}$$

$$a_{nj} = a_{nj}^{(1)} + a_{nj}^{(2)}; \quad a_{nj}^{(1)} = \frac{1}{\mu_2}\int_{-1}^{1} [u_n(x)\tau_j(x) + u_j(x)\tau_n(x)]\,dx$$

$$a_{nj}^{(2)} = \frac{1}{\mu_1\mu_2}\int_{\Omega} \{[k_1\varkappa_2^2\sigma_{yn}(\alpha)\sigma_{yj}(\alpha)\,\mathrm{sh}\,k_1H\,\mathrm{sh}\,k_2H -$$
$$k_2\varkappa_2^2\tau_n(\alpha)\tau_j(\alpha)\,\mathrm{ch}\,k_2H\,\mathrm{ch}\,k_1H +$$
$$i\alpha\Phi_1(\alpha, 0)[\tau_n(\alpha)\sigma_{yj}(\alpha) + \sigma_{yn}(\alpha)\tau_j(\alpha)]\}\frac{d\alpha}{\Delta(\alpha)}$$

$$b_j = -A\frac{\sqrt{2\pi}}{\mu_2}\left[\frac{i\alpha_0}{k_1^\circ}\Phi_1(\alpha_0, 0)\tau_j(\alpha_0) + \varkappa_2^2\,\mathrm{sh}\,k_2^\circ H\,\mathrm{sh}\,k_1^\circ H\sigma_{yj}(\alpha_0)\right]$$

$$\sigma_{yn}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\sigma_{yn}(x)\cos\alpha x\,dx,$$

$$\tau_n(\alpha) = -\frac{i}{\sqrt{2\pi}}\int_{-1}^{1}\tau_n(x)\sin\alpha x\,dx$$

The parameters of the Rayleigh-Lamb incident wave (amplitude A and wave number α_0) occur on the right-hand side of system (9).

We will obtain the total displacement field for $y = 0$, including both the symmetrical and antisymmetrical parts. The symmetrical part is specified by relations (3). We will change from integration over α to integration in the complex plane $\zeta = \alpha + i\eta$. When $x < -1$, closing the contour Ω in the lower part of the complex plane and calculating the residues at the poles, we obtain

$$u_1(x, 0) = \sum_{j=1}^{m} D_j \exp(-i\alpha_j x) + \sum_{k=1}^{\infty} E_k \exp(i\zeta_k x) + u_0(x, 0) \tag{10}$$

Here α_j are the real roots and ζ_k are the complex roots of Eq.(2), and D_j and E_k are constants which depend on all the parameters of the problem. The wave field contains a finite number of refracted waves travelling in the negative direction of the x axis, and an infinite number of non-uniform waves with attenuating amplitude. The number of travelling waves is equal to the number of poles of the integrand in (3). The phase velocities of the waves are determined by the elastic constants of the horizontal strip.

When $x > 1$ the wave field contains the same wave as in (10), but propagating in the positive direction of the x axis.

In the region $-1 \leqslant x \leqslant 0$ the displacement field can be represented in the form

$$u_1(x, 0) = \sum_{j=1}^{m}(F_j\cos\alpha_j x + G_j\sin\alpha_j x) + \sum_{j=1}^{m} H_j\exp(-i\alpha_j x) + \tag{11}$$
$$\sum_{k=1}^{\infty}[K_k\exp(i\zeta_k x) + L_k\exp(-i\zeta_k x)] + u_0(x, 0)$$

Here the first sum represents standing waves, and the second represents waves travelling in the negative direction of the x axis. The phase velocities of these waves are determined by the real roots of Eq.(2). The third sum represents attenuating waves, corresponding to the complex roots. The amplitudes of the waves in (11) are determined by all the parameters of the problem.

When $0 \leqslant x \leqslant 1$ the nature of the wave field is the same as in (11), with the exception of the travelling waves, which in this case change their direction of propagation.

At the corner points of the coupling region there is a singularity in the stressed state /6/. In polar coordinates ρ, φ with centre at the corner point, the stresses have the form

$$\sigma_\varphi \sim \rho^{\gamma-1}, \; \tau_{\rho\varphi} \sim \rho^{\gamma-1}, \; \rho \to 0 \tag{12}$$

where γ is found from the equation

$$\mu_2^2(1-\nu_1)^2(\sin^2{}^1/_2\gamma\pi - \gamma^2) + \mu_1^2(1-\nu_2)^2\sin^2\gamma\pi +$$
$${}^1/_4(\nu_1-\nu_2)^2(\sin^2{}^1/_2\gamma\pi - \gamma^2)\sin^2\gamma\pi + 2\mu_1\mu_2(1-\nu_2)(1-$$
$$\nu_1)\sin\gamma\pi\sin{}^1/_2\gamma\pi\cos{}^3/_2\gamma\pi + \mu_2(\mu_1-\mu_2)(1-\nu_1)(\sin^2{}^1/_2\gamma\pi -$$
$$\gamma^2)\sin^2\gamma\pi + \mu_1(\mu_2-\mu_1)(1-\nu_2)\sin^2\gamma\pi\sin^2{}^1/_2\gamma\pi = 0, 0 < \gamma < 1$$

Using representation (4) we can obtain

$$\sigma_{y2}(x, y) = \sum_{n=1}^{\infty} C_n \sigma_{yn}(x) \exp(i\beta_n y)$$

where $\sigma_{yn}(x)$ are the stresses corresponding to the uniform solutions. The asymptotic behaviour of $\sigma_{yn}(x)$ for large n is determined by the well-known asymptotic form of the roots of the dispersion equation /7, 8/

$$\beta_n \sim i\pi(n - {}^1/_4) + {}^1/_2 \ln \pi (4n - 1) + O(n^{-1} \ln n) \tag{13}$$

Using (13) and transferring to polar coordinates, we obtain that the behaviour of the stresses as $\rho \to 0$ is determined by the series

$$\sigma = (\sigma_\varphi, \tau_{\rho,\varphi}) \sim \sum_{n=N}^{\infty} C_n (-1)^n n^{-1/2} e^{-nz}, \quad N \gg 1 \tag{14}$$
$$z = \pi\rho \cos\varphi, \quad z \to 0, \quad \varphi \in [0, \pi/2]$$

It follows from (14) and (12) that the conditions at the corner point dictate the asymptotic behaviour of the constants C_n as $n \to \infty$. By finding the constants C_n in the form $C_n \sim C_0(-1)^n n^{\beta_0}$, where β_0 is to determined, we obtain from (14)

$$\sigma \sim \sum_{n=1}^{\infty} n^{\beta_0 - 1/2} e^{-nz}$$

We will use the result from /9/. If

$$f(z) = \sum_{n=1}^{\infty} n^q e^{-nz}, \quad z > 0, \quad -1 < q < 0$$

then the estimate $f(z) \sim z^{-(q+1)}, z \to +0$ holds. Hence, it follows from (12) that $\beta_0 = {}^1/_2 - \gamma$. Therefore, the choice of the asymptotic behaviour of the constants in the form

$$C_n = C_0(-1)^n n^{1/2 - \gamma}, \quad n \to \infty \tag{15}$$

ensures the required singularity of the stresses at the corner points of the contact (12). The infinite system (9), taking into account the specified asymptotic behaviour of C_n (15), reduces to the finite system

$$C_j S_j + \sum_{n=1}^{N_1} C_n a_{nj} + C_0 \sum_{n=N_1+1}^{N_2} (-1)^n n^{1/2 - \gamma} a_{nj} = b_j$$
$$j = 1, 2, \ldots, N_1 + 1; \quad N_2 \gg N_1$$

The improper integrals in a_{nj} can be evaluated by the methods of the theory of residues.

The above approach can be applied to the case of the oscillations of an elastic half-strip coupled at one end to an elastic half-plane, when subjected to a travelling Rayleigh wave. The overall form of the solution in this case does not differ from that described above. Problem (1) for an infinite strip reduces to the problem of the oscillations of a half-strip with load specified on the boundary and a Rayleigh wave propagating in the positive direction of the x axis. The improper integrals occurring in the solution can be found by the effective method described in /10/. The displacement field when $y = 0$ consists of travelling and scattered Rayleigh waves and attenuating longitudinal and transverse waves. In the contact region standing waves are added to the wave field, corresponding to the uniform solutions for the strip.

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REFERENCES

1. BABESHKO V.A. and PEL'TS S.P., Oscillations of a plate on a elastic layer, Izv. Akad. Nauk SSSR, MTT, 1, 1976.
2. PEL'TS S.P. and TSVETYANSKII V.L., Excitation of waves by a vibrating cylinder in a layer, Izv. Akad. Nauk SSSR, MTT, 4, 1982.
3. PEL'TS S.P., Vibrations of a cylinder on an elastic layer, partially coupled to a rigid base, PMM, 47, 5, 1983.
4. VOROVICH I.I. and BABESHKO V.A., Dynamic Mixed Problems of the Theory of Elasticity for Non-classical Regions, Moscow, Nauka, 1979.
5. ZIL'BERGLEIT A.S. and NULLER B.M., Generalized orthogonality of uniform solutions in dynamic problems of the theory of elasticity, Dokl. Akad. Nauk SSSR, 234, 2, 1977.
6. AKSENTYAN O.K. Features of the stress-deformation state of a plate in the neighbourhood of a rib, PMM, 31, 1, 1967.

7. ZLATIN A.N., The root of certain transcendental equations encountered in the theory of elasticity, Prikl. Mekhanika, 16, 12, 1980.
8. VOROVICH I.I., ALEKSANDROV V.M. and BABESHKO V.A., Non-classical Mixed Problems in the Theory of Elasticity, Moscow, Nauka, 1974.
9. MITTRA R. and LI S., Analytical Methods of the Theory of Waveguides, Moscow, Mir, 1974.
10. ALEKSANDROV V.M. and BURYAK V.G., Some dynamic mixed problems of the theory of elasticity, PMM, 42, 1, 1978.

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AVERAGING IN PROBLEMS OF THE BENDING AND OSCILLATION OF STRESSED INHOMOGENEOUS PLATES*

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A method for describing on the average the bending and oscillation of strongly inhomogeneous plates, stressed in their plane, is proposed. A problem that arises in various fields of engineering differs from those considered in /2-4/ in that the operators are not known a priori to be of fixed sign.

1. The bending of an inhomogeneous stressed plate. We consider a plate with irregular thickness of irregular elastic constants (ribbed or composition). Let forces be applied to the plate that produce in its plane a stressed state $\sigma_{ij}^\varepsilon(\mathbf{x})$ (the parameter ε characterizes the degree of irregularity). In the context of the Kirchhoff-Love hypothesis, the equation of equilibrium may be written as ($w^\varepsilon(\mathbf{x})$ is the normal bending of the plate) /1, 5/

$$-L_\varepsilon w^\varepsilon + M_\varepsilon w^\varepsilon \equiv [D^\varepsilon(w^\varepsilon_{,11} + \nu^\varepsilon w^\varepsilon_{,22})]_{,11} + 2[D^\varepsilon(1-\nu^\varepsilon)w^\varepsilon_{,12}]_{,12} + [D^\varepsilon(w^\varepsilon_{,22} + \nu^\varepsilon w^\varepsilon_{,11})]_{,22} - [\sigma_{11}^\varepsilon w^\varepsilon_{,1}]_{,1} - [\sigma_{12}^\varepsilon w^\varepsilon]_{,2} - [\sigma_{21}^\varepsilon w^\varepsilon_{,2}]_{,1} - [\sigma_{22}^\varepsilon w^\varepsilon_{,2}]_{,2} = f(\mathbf{x}) \tag{1.1}$$

The flexural rigidity $D^\varepsilon(\mathbf{x})$ and Poisson's ratio $\nu^\varepsilon(\mathbf{x})$ (we consider locally isotropic plates) depend on the space variable $\mathbf{x} \in Q$; $Q \subset R^2$ is the bounded domain occupied by the plate. As the dependence of D^ε, ν^ε on $\mathbf{x}$, we take /2, 6, 7/ $D^\varepsilon(\mathbf{x}) = D(\mathbf{x}/\varepsilon)$, $\nu^\varepsilon(\mathbf{x}) = \nu(\mathbf{x}/\varepsilon)$, where the functions $D(\mathbf{y})$, $\nu(\mathbf{y})$ have the characteristic size of oscillation equal to unity. The stresses $\sigma_{ij}^\varepsilon(\mathbf{x})$ in the plane of the plate are also functions of $\mathbf{x}$ with the characteristic size of oscillation equal to the characteristic size of the irregularity ε. For $\varepsilon \ll 1$, i.e., in the case of strongly irregular plates, in order to describe the bending and loss of stability we use /2-4, 8/ the asymptotic method of homogenization /6, 7/.

Problem (1.1) will be studied asymptotically as $\varepsilon \to 0$ with the proviso that the plate edges are rigidly clamped (we know /1, 2/ that this is equivalent to considering (1.1) in functional space $H_0^2(Q)$ /9, 10/). We consider the problem in the abstract statement. Given the sequences of linear selfadjoint operators, bounded uniformly as $\varepsilon \to 0$,

$$L_\varepsilon, L: H_0^2(Q) \to H^{-2}(Q);\quad M_\varepsilon, M: H_0^k(Q) \to H^{-k}(Q),\quad 0 \leqslant k < 2 \tag{1.2}$$

(for the definition of a space of type $H^\alpha(Q)$ see e.g., /9, 10/). The operator $-L_\varepsilon$ is the sum of the first three terms of the left-hand side of (1.1), while $-M_\varepsilon$ is the sum of the remaining terms, which describe the influence of the stresses in the plane of the plate on its normal bending. Let the operators L_ε and L be positive definite: there exists $c > 0$, independent of $\varepsilon \to 0$, such that $\langle L_\varepsilon u, u\rangle_2 \geqslant c\|u\|_2^2$ for any $u \in H_0^2(Q)$ ($\langle\cdot,\cdot\rangle_k$, $\|\cdot\|_k$ is the operation of pairing and norming in $H_0^k(Q)$ /9/).

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